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# Exact evolution of the generalized damped harmonic oscillator

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**Abstract.** In this paper we use the algebraic and the invariant method to study the time-dependent damped harmonic oscillator from classical and quantum points of view. The solution of the classical equation of motion and the wavefunction solving the time-dependent Schrödinger equation are found explicitly. We show that the original time-dependent quantum-mechanical problem is completely related to the well known time-independent harmonic oscillator. In addition, we elucidate the intimate connection between the damped harmonic oscillator (DHO) and the generalized harmonic oscillator (GHO). More importantly, the evolution of the states of the DHO cannot be cyclic, in contradistinction with the states of the GHO. Explicit expressions for both the dynamical and the geometric angles and phases are deduced in the adiabatic limit. The coherent states describing the invariant-angle variables of the classical DHO are constructed; they allow us to recover the classical evolution and invariant from the quantum evolution.

## 1. Introduction

In classical mechanics, the motion with friction can be described by the following Newtonian equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}(m\dot{q}) = -2\gamma\dot{q} - kq. \tag{1.1}$$

The first term on the right-hand side is a dissipative frictional force proportional to velocity, where $\gamma$ is a friction constant. The second term is a conservative external force which can be derived from a harmonic potential $kq^2/2$.

Two cases have been considered: (a) $m, k, \gamma > 0$ are constant [1–9], and (b) $\gamma = 0$, $m$ and $k$ are time dependent [10–12], from quantum or classical points of view. The general case, which we study in this paper, has a time-dependent mass, a time-dependent friction 'constant' and a time-dependent 'spring' constant. Equation (1.1) can be obtained from the generalized Caldirola–Kanai Hamiltonian,

$$H(\vec{\nu}(t)) = \frac{1}{2}\left(\frac{1}{m(t)}\,\mathrm{e}^{-2\int_0^t \lambda(s)\,\mathrm{d}s}\,p^2 + m(t)\,\omega_0^2(t)\,\mathrm{e}^{2\int_0^t \lambda(s)\,\mathrm{d}s}\,q^2\right) \tag{1.2}$$

where $\vec{\nu}(t)$ denotes the set of arbitrary time-dependent parameters $m(t)$, $\lambda(t) = \gamma(t)/m(t)$, $\omega_0^2(t) = k(t)/m(t)$ and $q, p$ are the canonical coordinates. For $\lambda = 0$ we recognize this as the Hamiltonian for a simple harmonic oscillator with time-dependent mass and frequency. In general, the physical system described by the Hamiltonian (1.2) represents the motion of a particle with a time-dependent mass $m(t)$, bound to a spring with time-dependent Hooke 'constant' $k(t) = m(t)\omega_0^2(t)$ and submitted to a frictional force with a time-dependent friction 'constant' $\gamma(t) = m(t)\lambda(t)$.

The interpretation of the Caldirola–Kanai Hamiltonian [1, 2] as describing damping in quantum mechanics, has been criticized [9] because of the apparent violation of the uncertainty principle. The uncertainty principle, in terms of the canonical momentum is always satisfied but in terms of the kinetic momentum, it is not, in general, satisfied [13]. In this paper the phenomenological friction 'constant' generalized to be time dependent can consequently be chosen so that the uncertainty principle, in terms of the kinetic momentum, is always satisfied.

The purpose of this paper is twofold. First, we want to derive the invariant and calculate the solution of the classical and quantum evolution of the system described by the generalized Caldirola–Kanai Hamiltonian (1.2). We deduce, in particular, the dynamical and geometric character of the angles and phases in the adiabatic limit. Second, we want to construct the invariant angle coherent states [14, 15] of the damped harmonic oscillator (DHO) to get a classical evolution from the quantum evolution expressed at the level of these coherent states in the classical limit. Our approach is much simpler, hence it deserves attention.

This paper is organized as follows. In section 2 we derive the invariant of the generalized damped oscillator. In section 3, we find the exact solution of the classical DHO. We obtain the evolved wavefunction of a DHO in the quantum case, in section 4. In section 5, we introduce the 'invariant-angle' coherent states [14, 15] relative to the DHO. They are characterized by a complex number $\alpha = \sqrt{I/\hbar}\,\mathrm{e}^{-\mathrm{i}\theta}$ the evolution of which describes the dynamics of the classical invariant $I$ and angle $\theta$ variables. In the concluding section, we use the adiabatic limit of the results obtained in sections 3 and 4 to find the Hannay angle [16] and Berry phase [17] of a DHO. We end with a comparative study between the DHO and the generalized harmonic oscillator (GHO).

## 2. Derivation of classical invariant

For the construction of an exact invariant for the classical dynamical systems described by the time-dependent Hamiltonian (1.2), we use the Lie algebraic approach, introducing the basis $T_1 = p^2/2,\ T_2 = pq,\ T_3 = q^2/2$ which forms a finite algebra with the following Poisson brackets:

$$\{T_1, T_2\} = -2T_1 \qquad \{T_2, T_3\} = -2T_3 \qquad \{T_3, T_1\} = T_2. \tag{2.1}$$

Now, we look for the generalized invariant of the form

$$I = \sum_r \mu_r(t) T_r \tag{2.2}$$

and by means of

$$\frac{\partial I}{\partial t} = \{I, H\}_{qp} \tag{2.3}$$

and comparison of coefficients, a system of first-order linear differential equations for the unknown $\mu_r$ in (2.2) is obtained

$$\begin{aligned} \dot{\mu}_1 &= -2\frac{\mu_2}{m}\mathrm{e}^{-2\int_0^t \lambda(t')\,\mathrm{d}t'} \\ \dot{\mu}_2 &= \omega_0^2 m\mu_1 \mathrm{e}^{2\int_0^t \lambda(t')\,\mathrm{d}t'} - \frac{\mu_3}{m}\mathrm{e}^{-2\int_0^t \lambda(t')\,\mathrm{d}t'} \\ \dot{\mu}_3 &= 2\omega_0^2 m\mu_2 \mathrm{e}^{2\int_0^t \lambda(t')\,\mathrm{d}t'} \end{aligned} \tag{2.4}$$

which can be simplified by setting

$$\mu_1(t) = \frac{\sigma^2(t)}{m(t)}\,\mathrm{e}^{-2\int_0^t \lambda(t')\,\mathrm{d}t'} \tag{2.5}$$

to give

$$\ddot{\sigma} + \Omega^2(t)\sigma = \frac{1}{\sigma^3} \tag{2.6}$$

where

$$\Omega^2(t) = \omega_0^2(t) - \lambda^2(t) - \left(\frac{\dot{m}(t)}{m(t)}\right)\lambda(t) - \dot{\lambda}(t) - \frac{1}{2}\left(\frac{\ddot{m}(t)}{m(t)}\right) + \frac{1}{4}\left(\frac{\dot{m}(t)}{m(t)}\right)^2 \tag{2.7}$$

which is, of course, the classical frequency. The solution to equation (2.6) is, in fact, crucial for solving the exact classical angles and quantum phases. Thus, the invariant can be written in the form ($m\dot{q} = p\mathrm{e}^{-2\int_0^t \lambda(t')\,\mathrm{d}t'}$)

$$I = \frac{1}{2}m\sigma^2\left[\left(\frac{q}{\sigma^2}\right)^2 + \left(\dot{q} + \left(\lambda + \frac{\dot{m}}{2m} - \frac{\dot{\sigma}}{\sigma}\right)q\right)^2\right]\mathrm{e}^{2\int_0^t \lambda(t')\,\mathrm{d}t'}. \tag{2.8}$$

We note that this invariant generalizes the quantity $m\omega^{-1}(\omega^2 q^2 + (\dot{q} + \lambda q)^2)\mathrm{e}^{2\lambda t}$, which is an integral of motion for fixed parameters (i.e. $\omega_0, m, \lambda$ are constants). It also coincides, in the absence of damping, with the invariant of the harmonic oscillator with time-dependent mass and frequency.

## 3. Exact solution of the classical damped oscillator

The existence of the invariant $I(p, q, \vec{\mu}(t))$ implies that the evolution of the angle variables can be determined by making a time-dependent canonical transformation of the phase space variables to invariant-angle variables. Transformation to invariant-angle variables is effected by the generating function

$$S(q, I, t) = \int^q \mathrm{d}q'\, p(q', I, t) = -\frac{m}{2}(\lambda + \dot{m}/2m - \dot{\sigma}/\sigma)\,\mathrm{e}^{2\int_0^t \lambda(t')\,\mathrm{d}t'}q^2$$
$$+ I\left\{\cos^{-1}\left(\frac{\sqrt{m}\,\mathrm{e}^{\int_0^t \lambda(t')\,\mathrm{d}t'}}{\sigma\sqrt{2I}}q\right) - \frac{\sqrt{m}\,\mathrm{e}^{\int_0^t \lambda(t')\,\mathrm{d}t'}}{\sigma\sqrt{2I}}q\sqrt{1 - \frac{m\,\mathrm{e}^{2\int_0^t \lambda(t')\,\mathrm{d}t'}}{\sigma^2 2I}q^2}\right\} \tag{3.1}$$

which gives

$$p = \frac{\partial S}{\partial q} = -\sqrt{2I}\mathrm{e}^{\int_0^t \lambda(t')\,\mathrm{d}t'}\frac{\sqrt{m}}{\sigma}\sqrt{1 - \frac{\sqrt{m}\,\mathrm{e}^{2\int_0^t \lambda(t')\,\mathrm{d}t'}}{\sigma^2 2I}q^2} - m\left(\lambda + \frac{\dot{m}}{2m} - \frac{\dot{\sigma}}{\sigma}\right)\mathrm{e}^{2\int_0^t \lambda(t')\,\mathrm{d}t'}q \tag{3.2}$$

and

$$\theta = \frac{\partial S}{\partial I} = \cos^{-1}\left(\frac{\sqrt{m}\,\mathrm{e}^{\int_0^t \lambda(t')\,\mathrm{d}t'}}{\sigma\sqrt{2I}}q\right) \tag{3.3}$$

and the new 'Hamiltonian'

$$K \equiv H + \frac{\partial S}{\partial t} = \frac{I}{\sigma^2} \tag{3.4}$$

where $K$ is a function of $I$ only, and the dynamical problem is considered to be essentially solved.

The obtained solution $\theta(t)$, from Hamilton's equation $\dot{\theta} = \partial K/\partial I$, is

$$\theta(t) = \int_0^t \frac{1}{\sigma^2(t')}\,\mathrm{d}t' + \theta_0 \tag{3.5}$$

(where $\theta_0$ is an arbitrary constant of integration). Then, the exact solution for $q$ is

$$q = \sqrt{\frac{2I}{m}}\,\mathrm{e}^{-\int_0^t \lambda(t')\,\mathrm{d}t'}\sigma\cos\left(\int_0^t \frac{1}{\sigma^2(t')}\,\mathrm{d}t' + \theta_0\right) \tag{3.6}$$

We observe that this solution generalizes the solution for fixed values of the parameters.

## 4. Exact solution: a quantum point of view

In the quantum theory of (1.2) the canonical coordinates $q$ and $p$ become quantum mechanical operators $q$ and $p = -\mathrm{i}\hbar\partial/\partial q$; the auxiliary function $\sigma$ remains a $c$-number. The invariant $I(t)$, equation (2.8), is a constant Hermitian operator satisfying $\partial I/\partial t = \mathrm{i}\hbar[I, H]$.

According to the Lewis–Riesenfeld theory [19], given a physical system that contains an invariant operator $I(t)$, the following results can be obtained:

(a) its eigenvalues $\beta_n$ are time independent,

$$I\psi_n(q,t) = \beta_n\psi_n(q,t) \tag{4.1}$$

(b) its eigenfunctions $\psi_n(q,t)$ depend on time.

Multiplied by suitable phases such as $\exp[\mathrm{i}\varphi_n(t)]$, the $\varphi_n(t)$ verifying

$$\hbar\frac{\mathrm{d}\varphi_n(t)}{\mathrm{d}t} = \langle\psi_n|\mathrm{i}\hbar\frac{\partial}{\partial t} - H|\psi_n\rangle \tag{4.2}$$

then, the wavefunctions $\Psi_n(q,t) = \exp[\mathrm{i}\varphi_n(t)]\psi_n(q,t)$ evolve according to the time-dependent Schrödinger equation. The general solution can then be written as

$$\Psi(q,t) = \sum_n C_n\Psi_n(q,t) \tag{4.3}$$

where the $C_n$ are arbitrary constant coefficients fixed by the initial conditions of the physical system.

The key point of our analysis is to perform the time-dependent unitary transformation $\phi_n(q) = \mathcal{U}(t)\psi_n(q,t)$, where

$$\mathcal{U}(t) = V(t)\,U(t) = \exp\left(-\frac{\mathrm{i}}{2\hbar}\left(\int\lambda(t')\,\mathrm{d}t' - \ln\frac{\sigma}{\sqrt{m}}\right)(pq + qp)\right)$$
$$\times\exp\left(\frac{\mathrm{i}}{2\hbar}m\left(\lambda + \dot{m}/2m - \dot{\sigma}/\sigma\right)\mathrm{e}^{2\int_0^t\lambda(t')\,\mathrm{d}t'}q^2\right). \tag{4.4}$$

The operator $I$ changes into $I' = \mathcal{U}I\mathcal{U}^{-1}$. The operator eigenvalue equation (4.1) becomes

$$I'\phi_n(q) = \frac{1}{2}\left[-\hbar^2\frac{\partial^2}{\partial q^2} + q^2\right]\phi_n(q) = \beta_n\phi_n(q). \tag{4.5}$$

We note that the new eigenvalue problem (4.5) is a usual time-independent Schrödinger equation for the harmonic oscillator. This fortuitous elimination of the time enables us to treat the original time-dependent problem by ordinary well known time-independent harmonic oscillator theory. Thus the corresponding stationary solution $\phi_n(q)$ is of the form

$$\phi_n(q) = \left(\pi^{1/2}\hbar^{1/2}2^n n!\right)^{-1/2} \mathrm{e}^{-q^2/2\hbar} H_n(q/\hbar^{1/2}). \tag{4.6}$$

The constant eigenvalue $\beta_n$ is exactly given by $\beta_n = \hbar(n + \frac{1}{2})$, $n = 0, 1, \ldots,$ and $H_n$ is the usual Hermite polynomials of order $n$. The solution $\psi_n(q, t) = \mathcal{U}^{-1}\, \phi_n(q)$ of (4.1) is thus given by

$$\psi_n(q, t) = \frac{m^{1/4}\, \mathrm{e}^{\int \mathrm{d}t\, \lambda/2}}{\sigma^{1/2}} \exp\left\{-\frac{\mathrm{i}}{2\hbar} m\big(\lambda + \dot{m}/2m - \dot{\sigma}/\sigma\big)\big(q\mathrm{e}^{\int \mathrm{d}t\, \lambda}\big)^2\right\} \phi_n\big(\mathrm{e}^{\int \lambda\, \mathrm{d}t}\sqrt{m}q/\sigma\big). \tag{4.7}$$

Note that the argument of the wavefunction depends on the damping phenomena. In equation (4.7) we have used an important property of the transformation $V(t)$ which, when it acts on a wavefunction in the $q$-representation, gives $V^{-1}(t)\, \phi_n(q) = (m^{1/4}\, \mathrm{e}^{\int \mathrm{d}t\, \lambda/2}/\sigma^{1/2})$ $\phi_n(\mathrm{e}^{\int \lambda\, \mathrm{d}t}\sqrt{m}q/\sigma)$.

Now we are in a position to find the phases $\varphi_n(t)$ which satisfy equation (4.2). Substituting (4.7) into equation (4.2) and using the auxiliary equation (2.6) to eliminate $\omega_0(t)$ from $H$, we find

$$\hbar\varphi_n(t) = \int_0^t \mathrm{d}t'\, \langle\phi_n| - \frac{1}{\sigma^2} I' |\phi_n\rangle = -\hbar\big(n + \tfrac{1}{2}\big) \int_0^t \mathrm{d}t'\, \frac{1}{\sigma^2}. \tag{4.8}$$

Thus, the evolved wavefunction of the DHO is

$$\Psi_n(q, t) = \exp\left[-\mathrm{i}\big(n + \tfrac{1}{2}\big) \int_0^t \mathrm{d}t'\, \frac{1}{\sigma^2}\right] \psi_n(q, t). \tag{4.9}$$

Finally, we should make some remarks. (a) In the absence of damping and when $m$ is constant and $\omega_0$ is time dependent, our new wavefunction reduces to that obtained in [20], and to that of [11, 12] in the case where $m$ and $\omega_0$ are time dependent. (b) In the presence of damping and when $m$, $\omega_0$, $\lambda > 0$ are constant the wavefunction (4.7) also reduces to that of [3, 4, 10]. However, our result is more general because it includes a time-varying mass, friction and frequency.

## 5. Invariant-angle coherent states for the damped oscillator

The classical results established in sections 2 and 3 can be derived from the quantum analysis. Indeed, as explained in [14, 15], a simple way to provide a quantum description of the evolution of a classical system to obtain the classical evolution, is to study the evolution of 'invariant-angle' coherent states which are a natural generalization of the well known coherent states of the harmonic oscillator.

Defining the annihilation $a'$ and creation $a'^{+}$ operators

$$a' = \frac{1}{(2\hbar)^{1/2}}\left(q + \mathrm{i}\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q}\right) \qquad a'^{+} = \frac{1}{(2\hbar)^{1/2}}\left(q - \mathrm{i}\frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial q}\right) \tag{5.1}$$

with the properties $a'\phi_n = \sqrt{n}\, \phi_{n-1}$, $a'^{+}\phi_n = \sqrt{n+1}\, \phi_{n+1}$, and $[a', a'^{+}] = 1$, the Invariant $I'$ (4.5), can be rewritten in the standard quadratic Hermitian form

$$I' = \hbar\big(a'^{+}a' + \tfrac{1}{2}\big). \tag{5.2}$$

As in the ordinary harmonic oscillator, the coherent states for $I'$ are defined by

$$\phi_\alpha(q) = \mathrm{e}^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \phi_n(q) \tag{5.3}$$

where $\alpha = |\alpha|\mathrm{e}^{-\mathrm{i}\theta_0}$ is a complex number. They are eigenfunctions of the lowering operator $a'$ with eigenvalue $\alpha$,

$$a'\phi_\alpha = \alpha\phi_\alpha. \tag{5.4}$$

The coherent states for the generalized time-dependent DHO are now obtained from $\phi_\alpha$ by application of the unitary transformation $\mathcal{U}$:

$$\psi_\alpha(q,t) = \mathcal{U}^{-1}\phi_\alpha = \mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\psi_n(q,t). \tag{5.5}$$

The eigenvalue equation (5.4) is mapped into

$$a\psi_\alpha(q,t) = \alpha\psi_\alpha(q,t) \tag{5.6}$$

where the transformed annihilation operator

$$\begin{aligned} a &= \mathcal{U}^{-1}a'\mathcal{U} \\ &= \sqrt{\frac{m}{2\hbar}}\left\{\frac{\mathrm{e}^{\int \mathrm{d}t\,\lambda}}{\sigma}q + \mathrm{i}\left(\frac{\sigma}{m}\mathrm{e}^{-\int \mathrm{d}t\,\lambda}\,p + \left((\lambda + \dot{m}/2\,m)\sigma - \dot{\sigma}\right)\mathrm{e}^{\int \mathrm{d}t\,\lambda}q\right)\right\} \end{aligned} \tag{5.7}$$

allows us to write the invariant $I$ in the form

$$I = \hbar\left(a^+a + \tfrac{1}{2}\right). \tag{5.8}$$

The eigenfunctions $\psi_\alpha(q,t)$ of the lowering operator $a$ are just the coherent states of the generalized time-dependent DHO, obtained in terms of the eigenfunctions of the invariant $I$. It is pointed out that these coherent states, constructed from the number states of the Lewis–Riesenfeld invariant, are the squeezed states [12]. During the quantum evolution of $\psi_\alpha(0)$ each eigenfunction $\psi_n(q,0)$ acquires the phase $\varphi_n(t)$ given by (4.8). The part independent of $n$ of $\varphi_n(t)$ brings in (5.5) a global phase factor. More interestingly, the part proportional to $n$ induces a change of the complex number $\alpha$ such that its modulus remains constant while its argument $\theta_0$ becomes $\theta_0 + \theta(t)$ with

$$\theta(t) = \int_0^t \frac{1}{\sigma^2}\,\mathrm{d}t'. \tag{5.9}$$

The quantum evolution of the state $\psi_\alpha(0)$ thus leads to the state

$$\Psi_{\alpha(t)} = \mathrm{e}^{-\frac{1}{2}\mathrm{i}\theta(t)}\psi_{\alpha(t)} \tag{5.10}$$

and amounts, up to a global inessential phase factor, to the evolution of the complex number $\alpha$:

$$\alpha \to \alpha(t) = |\alpha|\,\mathrm{e}^{-\mathrm{i}\theta(t)} \qquad (|\alpha(t)| = (|\alpha|\text{ constant}). \tag{5.11}$$

This describes the coherent state wavepackets whose position in phase space is specified by the shift in angle variable $\theta(t)$ associated with the invariant of the classical damped harmonic oscillator $\hbar|\alpha|^2 = \langle\psi_{\alpha(t)}|I|\psi_{\alpha(t)}\rangle$. However, we call them 'invariant-angle' coherent states because the complex number $\alpha$ can be related to the classical invariant-angle variables by $\alpha = \sqrt{I/\hbar}\,\mathrm{e}^{-\mathrm{i}\theta}$.

Then, deducing the expression of $q$ in terms of the $a$s from (5.8), one can calculate the average value of the coordinate operator in the states $\psi_{\alpha(t)}$,

$$\begin{aligned} \langle\psi_{\alpha(t)}|q|\psi_{\alpha(t)}\rangle &= \tfrac{1}{2}(2\hbar/m)^{1/2}\mathrm{e}^{-\int_0^t \lambda\,\mathrm{d}t'}\sigma\langle\psi_\alpha|a + a^+|\psi_\alpha\rangle \\ &= \left(\frac{2\hbar}{m}|\alpha|^2\right)^{1/2}\mathrm{e}^{-\int_0^t \lambda\,\mathrm{d}t'}\sigma\cos\left\{\theta_0 + \int_0^t \mathrm{d}t'\,\frac{1}{\sigma^2}\right\} \end{aligned} \tag{5.12}$$

which is exactly the solution (3.6) for a classical DHO with invariant $I = \hbar|\alpha|^2$. We can see that the Schrödinger property of coherent states, which give the classical motion, is satisfied for the coherent states $\psi_{\alpha(t)}$.

## 6. Conclusion

We now show that, in the adiabatic limit, the angle (3.5) and the phase (4.8) recover the Hannay angle and the Berry phase [22]. The adiabatic approximation can be obtained by ignoring terms with two or more time derivatives in equation (2.6) and taking for $\sigma(t)$ the adiabatic solution [15, 18, 21],

$$\frac{1}{\sigma^2} = \omega - \frac{\lambda}{2\omega}\left(\frac{\dot{m}}{m} + \frac{\dot{\lambda}}{\lambda}\right) \qquad \omega = \sqrt{\omega_0^2 - \lambda^2} \tag{6.1}$$

when this adiabatic expression is substituted into (3.5) or (4.8), we may obtain the total angle (or phase) accumulated in adiabatic evolution:

$$\theta(t) = \int_0^t \mathrm{d}t' \left(\omega - \frac{\lambda}{2\omega}\left(\frac{\dot{m}}{m} + \frac{\dot{\lambda}}{\lambda}\right)\right) \tag{6.2}$$

and

$$\varphi_n(t) = -(n + \tfrac{1}{2}) \int_0^t \mathrm{d}t' \left(\omega - \frac{\lambda}{2\omega}\left(\frac{\dot{m}}{m} + \frac{\dot{\lambda}}{\lambda}\right)\right) \tag{6.3}$$

where the first term is the dynamical angle (or phase) and the second term is the geometrical Hannay angle (or Berry phase) [22]. In this adiabatic limit, the invariant $I$ reduces to the adiabatic invariant of the classical DHO [22].

It is quite striking to note that the results obtained herein are similar to the case of the well known and extensively studied [16–18, 21–22] generalized harmonic oscillator (GHO)

$$H_{GHO} = \tfrac{1}{2}(Z(t)P^2 + 2Y(t)PQ + X(t)Q^2). \tag{6.4}$$

A natural 'mathematical' identification between the two systems (i.e. the DHO and the GHO, where the latter is studied in detail in [18]) can be established through a redefinition of the parameters of the GHO,

$$\begin{aligned} X(t) &= m(t)\omega_0^2(t) \exp\left\{2\int_0^t \lambda(s)\,\mathrm{d}s\right\} \qquad Y(t) = 0 \\ Z(t) &= \frac{1}{m(t)} \exp\left\{-2\int_0^t \lambda(s)\,\mathrm{d}s\right\}. \end{aligned} \tag{6.5}$$

Because of the total correspondence between the two systems (DHO and GHO) in the non-adiabatic case (i.e. when the parameters vary arbitrarily with time), one might be misled into thinking that DHO is a particular case of GHO, if one takes into consideration the aforementioned correspondence (6.5). Unfortunately, this method is not adapted in the adiabatic case (i.e. when the parameters vary slowly with time) where the results are well established [16, 17]. Consequently, one could think that the Berry phase for the DHO vanishes since the parameter of the mixed term $(pq + qp)$ does not appear in (1.2), which contradicts the results (6.1) established in the adiabatic limit. Therefore, the identification between the two systems would be more adequate if one makes a change of variables instead of parameters $P = p\mathrm{e}^{-\int^t \lambda(t')\,\mathrm{d}t'} = m\dot{q}\mathrm{e}^{\int^t \lambda(t')\,\mathrm{d}t'}$, $Q = q\mathrm{e}^{\int^t \lambda(t')\,\mathrm{d}t'}$ specified by the generating function $F(q, P, t) = qP\mathrm{e}^{\int^t \lambda(t')\,\mathrm{d}t'}$ in the classical case, or the unitary transformation $S(t) = \exp[\frac{\mathrm{i}}{2\hbar}\int^t \lambda(t')\,\mathrm{d}t'(pq + qp)]$ in the quantum case. This maps the DHO Hamiltonian onto the GHO Hamiltonian provided that one makes the following identification of the parameters:

$$X(t) = m(t)\omega_0^2(t) \qquad Y(t) = \lambda(t) \qquad Z(t) = \frac{1}{m(t)}. \tag{6.6}$$

This suggests that the two systems (the DHO and the GHO) are canonically equivalent (or unitarily equivalent).

Finally, another point to be raised has to do with the fact that in the cyclic case (i.e. when the parameters and the auxiliary function $\sigma(t)$ are periodic), the evolution of the states $\psi_n(t)$ of the damped harmonic oscillator cannot be cyclic, which is in contradiction with the states of the GHO. Indeed, the state $\psi_n(T)$ at time $T$, after the parameters had described a closed loop (taking the same values at time 0 and $T$), does not differ only by a phase factor from the initial state $\psi_n(0)$, as can be seen from the action of the operator $V(t)$ on a wavefunction $\phi_n$.

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